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The quasiclassical limit of the modified KP hierarchy

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Abstract. In the quasiclassical limit, the MKP hierarchy and its Lax representation turn into a modified Benney hierarchy and its Poisson representation. A Miura map is constructed, and shown to be canonical, from the modified Benney hierarchy into the unmodified one. The modified hierarchy is given both hydrodynamical and kinetic representations, and the Miura map is given a kinetic form. Explicit combinatorial formulae are proved for the infinite number of conserved densities of the modified Benney hierarchy.

1. Introduction

The KP hierarchy [1] has the form

$$\mathcal{L}_{,t} = [\mathcal{P}_+, \mathcal{L}] = [\mathcal{L}, \mathcal{P}_-] \tag{1.1}$$

where

$$\mathcal{L} = \xi + \sum_{i=0}^{\infty} A_i \xi^{-i-1} \tag{1.2}$$

is a Lax operator with the coefficients $A_i = A_i(x, t)$; $\mathcal{P}$ runs over the $\mathbb{Q}$-generators of positive ξ-degree of the centraliser $Z(\mathcal{L})$ of $\mathcal{L}$ in the associative ring of (algebraic) pseudodifferential operators (PDO) on the line:

$$\mathcal{P} = \mathcal{L}^m \qquad m \in \mathbb{N}. \tag{1.3}$$

The commutator [,] is taken in this ring with respect to the associative multiplication [2]

$$X \circ Y := \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n(X)}{\partial \xi^n} \partial^n(Y) \tag{1.4}$$

$$\partial := \partial/\partial x.$$

Finally, the + and − notation in formula (1.1) is defined by the rule: if $Q = \Sigma_l q_l \xi^l$ is a PDO then

$$Q_+ := \sum_{l \geq 0} q_l \xi^l \qquad Q_- := \sum_{l<0} q_l \xi^l. \tag{1.5}$$

The three most basic properties of the KP hierarchy (1.1) are as follows. All the flows, for various $\mathcal{P}$, (i) commute between themselves; (ii) have a common infinite set of conserved densities [1]

$$\{\mathrm{Res}(\mathcal{P}) | \mathcal{P} \in Z(\mathcal{L})\} \tag{1.6}$$

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where

$$\operatorname{Res}\left(\sum q_l \xi^l\right) := q_{-1} \tag{1.7}$$

and (iii) are Hamiltonian, with the Hamiltonian matrix $\bar{B}$ associated with the Lie algebra ${}_{0\leqslant}\mathrm{Diff}$ of left-differential operators (DO) on the line:

$$\bar{B} := {}_0\bar{B} = B({}_{0\leqslant}\mathrm{Diff}) \tag{1.8}$$

$${}_{0\leqslant}\mathrm{Diff} := \left\{ \sum_{n\geqslant 0} \partial^n g_n \,\middle|\, g_n \in K \right\} \tag{1.9}$$

where K is the basic differential $\mathbb{Q}$-algebra with a derivation ∂ (which can be thought of as $K = C^\infty\mathbb{R}^1$, $\partial = \partial/\partial x$) [3]. In the coordinates A_i, the matrix $\bar{B}$ has the matrix elements

$$\bar{B}_{ij} = \sum_n \left[\binom{j}{n} \partial^n A_{i+j-n} - \binom{i}{n} A_{i+j-n}(-\partial)^n \right]. \tag{1.10}$$

If we now take the quasiclassical limit of the KP hierarchy, i.e. change ∂ into $\varepsilon\partial$ in formula (1.4) and retain everywhere only the first order in ε terms, then from formula (1.4) we see that the commutator [,] in the ring of PDO turns into the Poisson bracket

$$\{X, Y\} = \frac{\partial X}{\partial \xi} \partial(Y) - \partial(X) \frac{\partial Y}{\partial \xi} \tag{1.11}$$

so that the Lax form (1.1) becomes

$$\mathscr{L}_{,t} = \{\mathscr{P}_+, \mathscr{L}\} = \{\mathscr{L}, \mathscr{P}_-\} \tag{1.12}$$

and the three basic properties of the KP hierarchy become the corresponding three basic properties of the resulting so-called Benney hierarchy [4], with the Hamiltonian matrix $\bar{B}$ (1.10) being replaced by its quasiclassical limit [5, 6]

$$B_{ij} = \partial_j A_{i+j-1} + i A_{i+j-1}\partial. \tag{1.13}$$

The first non-trivial term in the Benney hierarchy (1.12), for $\mathscr{P} = \mathscr{L}^2/2$, has the form

$$A_{i,t} = A_{i+1,x} + i A_{i-1} A_{0,x}. \tag{1.14}$$

This is the famous Benney system proper [7] and it results from two different physical systems. The first one describes long surface waves on a shallow fluid [7]:

$$u_{,t} = u u_{,x} + h_{,x} - u_{,y} \int_0^y \mathrm{d}y\, u_{,x} \qquad -\infty < x < \infty,\ u = u(x, y, t) \tag{1.15a}$$

$$h_{,t} = \left(\int_0^h \mathrm{d}y\, u \right)_{,x} \qquad 0 \leqslant y \leqslant h,\ h = h(x, t). \tag{1.15b}$$

The moments

$$A_i := \int_0^h \mathrm{d}y\, u^i \tag{1.16}$$

then satisfy the system (1.14). The second one [8] describes evolution of the one-particle distribution function $F(x, p, t)$ of a collisionless one-dimensional gas:

$$F_{,t} = pF_{,x} - \left(\int_{-\infty}^{\infty} \mathrm{d}p\, F\right)_{,x} F_{,p}. \tag{1.17}$$

The moments, defined now as

$$A := \int_{-\infty}^{\infty} \mathrm{d}p\, p^i F \tag{1.18}$$

then satisfy the same evolution system (1.14). The higher Benney flows (1.12) also have both hydrodynamic (1.16) [6] and kinetic (1.18) [9] representations, and this double representation property is available for *any* Hamiltonian system with the Hamiltonian matrix (1.13).

The modified KP (MKP) hierarchy has the form [3]

$$L_{,t} = [((P^{\dagger})_{\geqslant 1})^{\dagger}, L] = [L, ((P^{\dagger})_{\geqslant 0})^{\dagger}] \tag{1.19}$$

where

$$L = \xi + \sum_{i=0}^{\infty} a_i \xi^{-i} \qquad a_i = a_i(x, t). \tag{1.20}$$

P runs over the $\mathbb{Q}$-generators of positive ξ-degree of the centraliser $Z(L)$ of L in the ring of PDO:

$$P = L^m \qquad m \in \mathbb{N}. \tag{1.21}$$

'†' stands for 'adjoint':

$$(g\xi^l)^{\dagger} := (-\xi)^l g \qquad l \in \mathbb{Z}, g \in K \tag{1.22}$$

and if $Q = \Sigma_l\, q_l \xi^l$ is a PDO then

$$Q_{\leqslant \gamma} := \sum_{l \leqslant \gamma} q_l \xi^l \qquad \gamma \in \mathbb{Z} \tag{1.23}$$

and similar conventions apply for the notations $Q_{<\gamma}$, $Q_{>\gamma}$ and $Q_{\geqslant\gamma}$. The MKP hierarchy also has 'the three most basic properties': all its flows (i) commute between themselves; (ii) have an infinite common set of conserved densities $\{\mathrm{Res}(P) | P \in Z(L)\}$; and (iii) are Hamiltonian, with the Hamiltonian matrix $\bar{B}$ being the direct sum of the matrix

$$\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{1.24}$$

for the variables a_0 and a_1, and the matrix ${}_1\bar{B} = B({}_{1\leqslant}\mathrm{Diff})$ associated with the Lie algebra of left DO of order $\geqslant 1$, for the remaining variables $\{w_i := a_{i+2} | i \in \mathbb{Z}_+\}$:

$$({}_1\bar{B})_{ij} = \sum_n \left[\binom{j+1}{n} \partial^n w_{i+j+1-n} - \binom{i+1}{n} w_{i+j+1-n}(-\partial)^n\right] \tag{1.25}$$

$${}_{1\leqslant}\mathrm{Diff} = \left\{ \sum_{n \geqslant 0} \partial^{n+1} g_n \,\middle|\, g_n \in K \right\}. \tag{1.26}$$

The name 'modified KP' of the hierarchy (1.19) reflects the existence of the following Miura map [3] which sends the Lax representation of the MKP system ((1.19) and (1.21)) into the Lax representation of the KP system ((1.1) and (1.3)):

$$\xi+\sum_{i=0}^{\infty} A_i\xi^{-i-1}$$

$$=\mathscr{L}=\exp\left(\int a_0\,\mathrm{d}x\right)L\exp\left(-\int a_0\,\mathrm{d}x\right)$$

$$=\exp\left(\int a_0\,\mathrm{d}x\right)\left(\xi+\sum_{i=0}^{\infty} a_i\xi^{-i}\right)\exp\left(-\int a_0\,\mathrm{d}x\right) \tag{1.27a}$$

$$=\xi+\sum_{i=0}^{\infty} a_{i+1}(\xi-a_0)^{-i-1}=\xi+\sum_{i,n} a_{i+1}\binom{-i-1}{n}Q_n(-a_0)\xi^{-i-n} \tag{1.27b}$$

so that

$$A_i=\sum_{j+n=i} a_{j+1}\binom{-j-1}{n}Q_n(-a_0) \tag{1.28}$$

where

$$Q_n(-a_0):=(\partial-a_0)^n(1). \tag{1.29}$$

Taking the quasiclassical limit of the mKP hierarchy (1.19) we arrive at what can be called the 'modified Benney hierarchy':

$$L_{,t}=\{P_{\geqslant 1}, L\}=\{L, P_{\leqslant 0}\} \tag{1.30}$$

which is the subject of this paper. The main issue are as follows.

(i) The conjugation $L\to\exp(\int a_0\,\mathrm{d}x)L\exp(-\int a_0\,\mathrm{d}x)$ (1.27a) no longer makes sense in the quasiclassical setup, while the formal shift of ξ by $-a_0$ (1.27b) still does. Next section 2 is devoted to a proof that this shift maps the modified Benney hierarchy (1.30) into the Benney hierarchy (1.12).

(ii) The Miura map (1.28) is conjectured in [3] to be a canonical map between the Hamiltonian structure $\bar{b}$ ((1.24)$\oplus$(1.25)) of the MKP hierarchy and the Hamiltonian structure $\bar{B}$ (1.10) of the KP hierarchy. This conjecture is still unproven. In section 3 it will be shown that the quasiclassical version of this conjecture *is* true: the quasi-classical limit of the Miura map (1.28)

$$A_i=\sum_{j+n=i} a_{j+1}\binom{-j-1}{n}(-a_0)^n \qquad i\in\mathbb{Z}_+ \tag{1.31}$$

is a canonical map between the Hamiltonian structures

$$b=\left(\begin{array}{cc|c} 0 & \partial & \\ \partial & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & {}_1B \end{array}\right) \tag{1.32}$$

$$({}_1B)_{ij}:=\partial(j+1)w_{i+j}+(i+1)w_{i+j}\partial \tag{1.33}$$

and B (1.13).

(iii) The first non-trivial flow of the modified Benney hierarchy (1.30), with $P=L^2/2$, has the form

$$a_{i,t}=a_{i+1,x}+a_0a_{i,x}+ia_ia_{0,x} \qquad i\in\mathbb{Z}_+. \tag{1.34}$$

It is easy to check that this system has both the hydrodynamical representation

$$v_{,t}=\left(\tfrac{1}{2}v^2+v\int_0^{\bar h}\mathrm{d}y\,v\right)_{,x}-v_{,y}\int_0^y\mathrm{d}y\,v_{,x}\qquad -\infty<x<\infty,\ v=v(x,t) \tag{1.35a}$$

$$\bar h_{,t}=\left(\int_0^{\bar h}\mathrm{d}y\,v+\tfrac{1}{2}\bar h^2\right)_{,x}\qquad 0\leqslant y\leqslant \bar h,\ \bar h=\bar h(x,t) \tag{1.35b}$$

which generates (1.34) via the map

$$a_i=\int_0^{\bar h}\mathrm{d}y\,v^i\qquad i\in\mathbb{Z}_+ \tag{1.36}$$

and the kinetic representation:

$$f_{,t}=\left(p+\int_{-\infty}^{\infty}\mathrm{d}p\,f\right)f_{,x}-(pf)_{,p}\qquad f=f(x,p,t) \tag{1.37}$$

which generates (1.34) via the map

$$a_i=\int_{-\infty}^{\infty}\mathrm{d}p\,p^if\qquad i\in\mathbb{Z}_+. \tag{1.38}$$

In section 4 we show that *every* flow in the modified Benney hierarchy has such a double representation.

(iv) In section 5 we show that there exists a Miura map between the *kinetic equations* $f_{,t}=\ldots$ and $F_{,t}=\ldots$ which correspond to modified and unmodified Benney flows, such that the lift (1.38) and (1.18) of this map into the spaces of the moments a_i and A_i is given precisely by the Miura map (1.31). In fact, this kinetic Miura map is

$$F(x,p,t)=\left(p-\int_{-\infty}^{\infty}\mathrm{d}p\,f\right)f\left(x,p-\int_{-\infty}^{\infty}\mathrm{d}p\,f,t\right)=(p-a_0)f(x,p-a_0,t). \tag{1.39}$$

(v) Section 6 is devoted to explicit combinatorial formulae for the conserved densities $\{\mathrm{Res}(L^m)|m\in\mathbb{N}\}$ of the modified Benney hierarchy.

2. The Miura map

For the duration of this section we consider a slightly more general form of L and $\mathscr{L}$. Let

$$L=\xi^M+\sum_{i=0}^{\infty}a_i\xi^{M-1-i}\qquad M\in\mathbb{N}. \tag{2.1}$$

Consider the equations

$$L_{,t}=\{P_{\geqslant 1},L\}=\{L,P_{\leqslant 0}\} \tag{2.2}$$

where

$$P=L^{m/M}=\xi^m+\ldots\qquad m\in\mathbb{N} \tag{2.3}$$

and the Poisson bracket $\{\,,\}$ is the usual one on $T^*(\mathbb{R}^1)$ (1.11). Denote

$$\mathrm{ord}\left(\sum_l q_l\xi^l\right):=\max\{l|q_l\neq 0\}. \tag{2.4}$$

Since

$$\mathrm{ord}(\{X, Y\}) \leqslant \mathrm{ord}(X) + \mathrm{ord}(Y) - 1$$

from the second equality in (2.2) we see that

$$\mathrm{ord}(L_{,t}) = \mathrm{ord}(\{L, P_{\leqslant 0}\}) \leqslant M + 0 - 1 = M + 1.$$

Hence the motion equations (2.2) make sense. Finally, since $P = L^{m/M}$ and $\{\cdot, \cdot\}$ is a derivation with respect to each argument, we have

$$0 = \{P, L\} = \{P_{\geqslant 1} + P_{\leqslant 0}, L\} = \{P_{\geqslant 1}, L\} - \{L, P_{\leqslant 0}\}.$$

This shows that the second equality in (2.2) follows from the first one. In particular,

$$\begin{aligned} a_{0,t} &= \xi^{M-1}-\text{coefficient in } L_{,t} = \mathrm{Res}(L_{,t}\xi^{-M}) \text{ (by (2.2))} \\ &= \mathrm{Res}(\{L, P_{\leqslant 0}\}\xi^{-M}) = \mathrm{Res}(\{\xi^M, \mathrm{Res}(P\xi^{-1})\}\xi^{-M}) \\ &= M\partial[\mathrm{Res}(P\xi^{-1})]. \end{aligned} \tag{2.5}$$

Denote

$$\mathscr{C}_a := K[a_i^{(j)}] \qquad i, j \in \mathbb{Z}_+ \tag{2.6}$$

and let us make $\mathscr{C}_a$ into a differential $\mathbb{Q}$-algebra with a derivation ∂ by letting ∂ act on the polynomial generators of $\mathscr{C}_a$ by the rule

$$\partial(a_i^{(j)}) := a_i^{(j+1)}. \tag{2.7}$$

Pick

$$\rho \in C^\infty(\mathbb{R}^1) \tag{2.8}$$

and let

$$\varphi: C^\infty(T^*(\mathbb{R}^1)) \to C^\infty(T^*(\mathbb{R}^1)) \tag{2.9}$$

be the transformation

$$\varphi(x) = x \qquad \varphi(\xi) = \xi + \rho(x). \tag{2.10}$$

Since

$$\varphi(\mathrm{d}\xi \wedge \mathrm{d}x) = \mathrm{d}\xi \wedge \mathrm{d}x$$

φ is a canonical transformation. Hence

$$\varphi(\{X, Y\}) = \{\varphi(X), \varphi(Y)\}. \tag{2.11}$$

Since the equality (2.11) can be viewed as based on the commutation relations

$$\varphi \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} \varphi \tag{2.12}$$

$$\varphi \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x} - \rho_{,x} \frac{\partial}{\partial \xi}\right) \varphi \tag{2.13}$$

and these relations remain true when one considers $\rho \in \mathscr{C}_a$, and $\mathscr{C}_a((\xi^{-1}))$ instead of $C^\infty(T^*(\mathbb{R}^1))$, formula (2.11) remains true in these extended circumstances.

Theorem 2.1. Let

$$\rho = -\frac{1}{M} a_0. \tag{2.14}$$

Then

(a) the motion equations (2.2) imply the motion equations

$$[\varphi(L)]_{,t} = \{\varphi(P_{\geq 1}) + \operatorname{Res}(P\xi^{-1}), \varphi(L)\} \tag{2.15a}$$

$$= \{\varphi(L), \varphi(P_{\leq 0}) - \operatorname{Res}(P\xi^{-1})\} \tag{2.15b}$$

(b) if $P = L^{m/M}$ then, in formulae (2.15),

$$\varphi(P_{\geq 1}) + \operatorname{Res}(P\xi^{-1}) = ([\varphi(L)]^{m/M})_+ \tag{2.16}$$

$$\varphi(P_{\leq 0}) - \operatorname{Res}(P\xi^{-1}) = ([\varphi(L)^{m/M}])_-. \tag{2.17}$$

Remark 2.2. We have

$$\varphi(L) = \varphi(\xi^M + a_0\xi^{M-1} + \ldots) = (\xi+\rho)^M + a_0(\xi+\rho)^{M-1} + \ldots$$
$$= \xi^M + M\xi^{M-1}\rho + \ldots + a_0\xi^{M-1} + \ldots \text{(by (2.14)} = \xi^M + 0 \times \xi^{M-1} + \ldots. \tag{2.18}$$

Hence, we can identify

$$\varphi(L) = \xi^M + \sum_{i=0}^{\infty} A_i\xi^{M-2-i} =: \mathscr{L}. \tag{2.19}$$

Then formulae (2.15)-(2.17) imply that

$$\mathscr{L}_{,t} = \{\mathscr{P}_+, \mathscr{L}\} = \{\mathscr{L}, \mathscr{P}_-\} \tag{2.20}$$

with

$$\mathscr{P} = \mathscr{L}^{m/M}. \tag{2.21}$$

Thus, the Miura map φ (2.19) sends the modified system ((2.2) and (2.3)) into the unmodified one (2.20) and (2.21)).

Proof of theorem 2.1. (a) Suppose

$$L_{,t} = \{S, L\} \tag{2.22}$$

with some $S \in \mathscr{C}_a((\xi^{-1}))$. Considering ρ as an element in $\mathscr{C}_a$, we have

$$\frac{\partial}{\partial t}\varphi = \varphi\frac{\partial}{\partial t} + \rho_{,t}\varphi\frac{\partial}{\partial \xi}. \tag{2.23}$$

Thence

$$\begin{aligned}
[\varphi(L)]_{,t} &= \varphi(L_{,t}) + \rho_{,t}\varphi\left(\frac{\partial L}{\partial \xi}\right) && \text{(by (2.22) and (2.12))}\\
&= \varphi(\{S, L\}) + \rho_{,t}[\varphi(L)]_{,\xi} && \text{(by (2.11), (2.14) and (2.5))}\\
&= \{\varphi(S), \varphi(L)\} - [\varphi(L)]_{,\xi}\partial[\operatorname{Res}(P\xi^{-1})]\\
&= \{\varphi(S), \varphi(L)\} - \{\varphi(L), \operatorname{Res}(P\xi^{-1})\}\\
&= \{\varphi(S) + \operatorname{Res}(P\xi^{-1}), \varphi(L)\}.
\end{aligned} \tag{2.24}$$

From formula (2.2) we see that S is $P_{\geqslant 1}$ or $-P_{\leqslant 0}$, and formula (2.24) then yields both formulae (2.15).

(*b*) Since, by (2.4) and (2.10),

$$\varphi \circ \mathrm{ord} = \mathrm{ord} \circ \varphi, \tag{2.25}$$

we have, for $S = -P_{\leqslant 0}$ in (2.24),

$$\mathrm{ord}[-\varphi(P_{\leqslant 0}) + \mathrm{Res}(P\xi^{-1})] \leqslant -1 \tag{2.26a}$$

while for $S = P_{\geqslant 1}$,

$$\mathrm{ord}[\varphi(P_{\geqslant 1}) + \mathrm{Res}(P\xi^{-1})] \geqslant 0. \tag{2.26b}$$

Also

$$[\varphi(P_{\geqslant 1}) + \mathrm{Res}(P\xi^{-1})] - [-\varphi(P_{\leqslant 0}) + \mathrm{Res}(P\xi^{-1})]$$
$$= \varphi(P_{\geqslant 1}) + \varphi(P_{\leqslant 0}) = \varphi(P_{\geqslant 1} + P_{\leqslant 0}) = \varphi(P) = \varphi(L^{m/M}) = [\varphi(L)]^{m/M}.$$

Hence, by (2.26),

$$-\varphi(P_{\leqslant 0}) + \mathrm{Res}(P\xi^{-1}) = -([\varphi(L)]^{m/M})_-$$
$$\varphi(P_{\geqslant 1}) + \mathrm{Res}(P\xi^{-1}) = ([\varphi(L)]^{m/M})_+$$

as required.

3. The Miura map is canonical

Let

$$\mathscr{C}_A = K[A_i^{(j)}] \qquad i, j \in \mathbb{Z}_+ \tag{3.1}$$

be the differential $\mathbb{Q}$-algebra generated by the A_i. Let $\Phi \mathscr{C}_A \to \mathscr{C}_a$ be the differential homomorphism over K corresponding to the Miura map (1.31):

$$\Phi(A_i) = \sum_{j+n=i} a_{j+1} \binom{-j-1}{n} (-a_0)^n. \tag{3.2}$$

Theorem 3.1. The map Φ is canonical between the Hamiltonian structures B (1.13) in $\mathscr{C}_A$ and b (1.32) in $\mathscr{C}_a$.

Proof. Denote by $D(\Phi)$ the Fréchet derivative of Φ:

$$[D(\Phi)]_{ij} = \frac{\partial \Phi(A_i)}{\partial a_j}. \tag{3.3}$$

The canonical property of the map Φ is equivalent to the identity [10]:

$$D(\Phi) b D(\Phi)^{\dagger} = \Phi(B) \tag{3.4}$$

which we shall proceed to prove.

Since

$$\binom{-j-1}{n} = (-1)^n \binom{j+n}{n} \tag{3.5}$$

the map Φ (3.2) can be rewritten as

$$\Phi(A_i)=\sum_{j+n=i} a_{j+1}\binom{j+n}{n}(-1)^n(-a_0)^n=\sum_{j+n=i}\binom{i}{n}a_0^n a_{j+1}$$

$$=\sum_{n=0}^{i}\binom{i}{n}a_0^n a_{1+i-n}=a_0^i a_1+\sum_{n=0}^{i-1}\binom{i}{n}a_0^n w_{i-1-n}$$

$$=a_0^i a_1+\sum_{n=0}^{i-1}\binom{i}{n+1}w_n a_0^{i-1-n} \tag{3.6}$$

where

$$w_i := a_{i+2} \qquad i\in\mathbb{Z}_+. \tag{3.7}$$

Hence, the matrix elements of the matrix $D(\Phi)$ (3.3) are

$$\begin{array}{l} \\ \Phi(A_k) \end{array}\left(\begin{array}{c|c|c} a_0 & a_1 & w_i \\ ka_0^{k-1}a_1+\sum_{\alpha=0}^{k-1}\binom{k}{\alpha+1}(k-1-\alpha)w_\alpha a_0^{k-2-\alpha} & a_0^k & \binom{k}{i+1}a_0^{k-1-i} \end{array}\right). \tag{3.8}$$

Therefore, by formula (1.32), the matrix elements of the matrix $D(\Phi)b$ are

$$\begin{array}{ll} \Phi(A_k)-a_0 & a_0^k\partial \\ \Phi(A_k)-a_1 & [ka_0^{k-1}a_1+\sum_{\alpha=0}^{k-1}\binom{k}{\alpha+1}(k-1-\alpha)w_\alpha a_0^{k-2-\alpha}]\partial \\ \Phi(A_k)-w_j & \sum_i\binom{k}{i+1}a_0^{k-1-i}({}_1B)_{ij}. \end{array} \tag{3.9}$$

By formula (3.8), the matrix elements of the matrix $D(\Phi)^\dagger$ are

$$\begin{array}{ll} & \Phi(A_s) \\ \begin{array}{l} a_0 \\ a_1 \\ w_j \end{array} & \left(\begin{array}{c} sa_0^{s-1}a_1+\sum_{\beta=0}^{s-1}\binom{s}{\beta+1}(s-1-\beta)w_\beta a_0^{s-2-\beta} \\ a_0^s \\ \binom{s}{j+1}a_0^{s-1-j} \end{array}\right) \end{array}. \tag{3.10}$$

Hence, the $\Phi(A_k)-\Phi(A_s)$ matrix element of the matrix $D(\Phi)BD(\Phi)^\dagger$ on the LHS of formula (3.4) is

$$a_0^k\partial\left[sa_0^{s-1}a_1+\sum_{\beta=0}^{s-1}\binom{\beta}{s+1}(s-1-\beta)w_\beta a_0^{s-2-\beta}\right] \tag{3.11a}$$

$$+\left[ka_0^{k-1}a_1+\sum_{\alpha=0}^{k-1}\binom{k}{\alpha+1}(k-1-\alpha)w_\alpha a_0^{k-2-\alpha}\right]\partial a_0^s \tag{3.11b}$$

$$+\sum_{i,j}\binom{k}{i+1}\binom{s}{j+1}a_0^{k-1-i}[(i+1)w_{i+j}\partial+\partial(j+1)w_{i+j}]a_0^{s-1-j}. \tag{3.11c}$$

On the RHS of formula (3.4), the $A_k - A_s$ matrix element is, by formulae (1.13) and (3.6),

$$\Phi(kA_{k+s-1}\partial + \partial sA_{k+s-1})$$
$$= k\left[a_0^{k+s-1}a_1 + \sum_{\gamma=0}^{k+s-2}\binom{k+s-1}{\gamma+1}w_\gamma a_0^{k+s-2+\gamma}\right]\partial \tag{3.12a}$$
$$+\partial s\left[a_0^{k+s-1}a_1 + \sum_{\gamma=0}^{k+s-2}\binom{k+s-1}{\gamma+1}w_\gamma a_0^{k+s-2-\gamma}\right]. \tag{3.12b}$$

We have to show that (3.11) = (3.12). We start with the terms involving a_1: these are the first summands in (3.11a, b) and in (3.12a, b). Multiplying these terms, from the left by a_0^{-k}, and from the right by a_0^{-s}, we get the following identity to verify:

$$\partial sa_0^{-1}a_1 + ka_0^{-1}a_1\partial = ka_0^{-1}a_1(a_0^s\partial a_0^{-s}) + (a_0^{-k}\partial a_0^k)sa_0^{-1}a_1 \tag{3.13}$$

which is obviously true since

$$k(a_0^s\partial a_0^{-s}) = k\partial - ksa_0^{-1}a_0^{(1)} \qquad (a_0^{-k}\partial a_0^k)s = s\partial + ksa_0^{-1}a_0^{(1)}. \tag{3.14}$$

The remaining terms are all linear in the w. Picking out all the terms containing $w := w_\gamma$, for a fixed $\gamma \in \mathbb{Z}_+$, and multiplying each of these terms by a_0^k from the left, and by a_0^{-s} from the right, we arrive at the following identity to verify:

$$\partial\binom{s}{\gamma+1}(s-1-\gamma)wa_0^{-2-\gamma} + \binom{k}{\gamma+1}(k-1-\gamma)wa_0^{-2-\gamma}\partial$$
$$+\sum_{i+j=\gamma}\binom{k}{i+1}\binom{s}{j+1}a_0^{-1-i}[(i+1)w\partial + \partial(j+1)w]a_0^{-j-1} \tag{3.15L}$$
$$= k\binom{k+s-1}{\gamma+1}wa_0^{-2-\gamma}(a_0^s\partial a_0^{-s}) + (a_0^{-k}\partial a_0^k)s\binom{k+s-1}{\gamma+1}wa_0^{-2-\gamma} \text{ (by (3.14))}$$
$$= \binom{k+s-1}{\gamma+1}(kwa_0^{-2-\gamma}\partial + \partial swa_0^{-2-\gamma}). \tag{3.15R}$$

Our strategy in checking the operator identity (3.15) is, first, to collect terms linear in $w^{(1)}$; the remaining terms are proportional from the left to w, so that we can get rid of w. Afterwards, we repeat the separation procedure with the remaining a_0. So, terms proportional to $w^{(1)}$ amount to the identity

$$\binom{s}{\gamma+1}(s-1-\gamma)a_0^{-2-\gamma} + \sum_{i+j=\gamma}\binom{k}{i+1}\binom{s}{j+1}a_0^{-1-i}(j+1)a_0^{-j-1}$$
$$= \binom{k+s-1}{\gamma+1}sa_0^{-2-\gamma} \tag{3.16}$$

which, after being divided by $a_0^{-2-\gamma}$, becomes

$$\binom{s}{\gamma+1}(s-1-\gamma) + \sum_{i+j=\gamma}\binom{k}{i+1}\binom{s}{j+1}(j+1) = s\binom{k+s-1}{\gamma+1}. \tag{3.17}$$

To prove formula (3.17), we first transform the sum on the LHS:

$$\sum_{i+j=\gamma}\binom{k}{i+1}\binom{s}{j+1}(j+1) = \sum_{\substack{\alpha+\beta=\gamma+2\\ \alpha,\beta\neq 0}}\binom{k}{\alpha}\binom{s}{\beta}\beta$$
$$= \sum_{\alpha+\beta=\gamma+2}\binom{k}{\alpha}\binom{s}{\beta}\beta - \binom{s}{\gamma+2}(\gamma+2)$$

so that (3.17) becomes

$$\binom{s}{\gamma+1}(s-1-\gamma)-\binom{s}{\gamma+2}(\gamma+2) \tag{3.18a}$$

$$+\sum_{\alpha+\beta=\gamma+2}\binom{k}{\alpha}\binom{s}{\beta}\beta=s\binom{k+s-1}{\gamma+1}. \tag{3.18b}$$

This identity follows from the formulae

$$\binom{s}{n}(s-n)=\binom{s}{n+1}(n+1) \tag{3.19}$$

$$\sum_{\alpha+\beta=n}\binom{k}{\alpha}\binom{s}{\beta}\beta=s\binom{k+s-1}{n-1}. \tag{3.20}$$

Let us start with (3.19). It is true for $s=n$, and for $n>0$. For $s>n>0$

$$\binom{s}{n}(s-n)=\frac{s!}{n!(s-n)!}(s-n)=\frac{s!}{n!(s-n-1)!}$$

$$=(n+1)\frac{s!}{(n+1)!(s-n-1)!}=(n+1)\binom{s}{n+1}$$

as required. The identity (3.20) results from picking out the z^n coefficients in the identity

$$sz(1+z)^{s+k-1}=(1+z)^k sz(1+z)^{s-1}$$

$$=(1+z)^k z\frac{\mathrm{d}}{\mathrm{d}z}[(1+z)^s]$$

$$=\sum_{\alpha,\beta}\binom{k}{\alpha}z^{\alpha}z\frac{\mathrm{d}}{\mathrm{d}z}\left[\binom{s}{\beta}z^{\beta}\right]$$

$$=\sum_{\alpha,\beta}\binom{k}{\alpha}\binom{s}{\beta}\beta z^{\alpha+\beta}.$$

Thus, we can drop off the w terms from the identity (3.15). The remaining identity to be verified is

$$\partial\binom{s}{\gamma+1}(s-1-\gamma)a_0^{-2-\gamma}+\binom{k}{\gamma+1}(k-1-\gamma)a_0^{2-\gamma}\partial$$

$$+\sum_{i+1}\binom{k}{i+1}\binom{s}{j+1}(i+j+2)a_0^{-1-i}\partial a_0^{-1-j} \tag{3.21L}$$

$$=\binom{k+s-1}{\gamma-1}(ka_0^{-2-\gamma}\partial+\partial s a_0^{-2-\gamma}). \tag{3.21R}$$

Picking out the $a_0^{-3-\gamma}a_0^{(1)}$ terms in formula (3.21) we get

$$\binom{s}{\gamma+1}(s-1-\gamma)(-2-\gamma)+\sum_{i+j=\gamma}\binom{k}{i+1}\binom{s}{j+1}(\gamma+2)(-1-j)=\binom{k+s-1}{\gamma-1}s(-2-\gamma) \tag{3.22}$$

which is $(-2-\gamma)$ times the identity (3.17). Finally, the remaining a_0-free terms in formula (3.21) amount to

$$\binom{s}{\gamma+1}(s-1-\gamma)+\binom{k}{\gamma+1}(k-1-\gamma)+\sum_{i+j=\gamma}\binom{k}{i+1}\binom{s}{j+1}[(i+1)+(j+1)]$$
$$=\binom{k+s-1}{\gamma-1}(k+s). \tag{3.23}$$

This is the identity (3.17) added to (itself, with the indices (k and s) and (i and j) interchanged).

Remark 3.2. The map Φ thus sends any system of Hamiltonians in $\mathscr{C}_A$ commuting with respect to the Hamiltonian structure B (1.13) into a system of Hamiltonians in $\mathscr{C}_a$ commuting with respect to the Hamiltonian structure b (1.32). (A large number of commuting systems in $\mathscr{C}_A$ is given in [11].)

4. Hydrodynamical and kinetic representations

In this section we show that each modified Benney flow (1.30) has both a hydrodynamical and a kinetic representation.

First we rewrite the motion equations (1.30) in the ϖ language. Set

$$L^m=\sum_l q_l(m)\xi^l. \tag{4.1}$$

Then

$$\begin{aligned}L_{,t}&=\sum_{i=0}^{\infty}a_{i,t}\xi^{-i}=\{L,P_{\leqslant 0}\}=\{L,P_{\leqslant 0}\}_{\leqslant 0}\\&=\{P_{\geqslant 1},L\}_{\leqslant 0}=\{P_{\geqslant 1},L_{\leqslant 0}\}_{\leqslant 0}\\&=\left\{\sum_{r\geqslant 1}q_r(m)\xi^r,\sum_{j=0}^{\infty}a_j\xi^{-j}\right\}_{\leqslant 0}\\&=\left(\sum_{r\geqslant 1}\sum_{j\geqslant 0}\xi^{r-j-1}[rq_r(m)a_{j,x}+jq_r(m)_{,x}a_j]\right)_{\leqslant 0}\end{aligned}$$

so that, suppressing m from the notation $q_r(m)$, we obtain

$$a_{i,t}=\sum_{j\geqslant 0}[(j+1)q_{j+1}a_{i+j,x}+(i+j)q_{j+1,x}a_{i+j}] \tag{4.2a}$$
$$=\sum_{j\geqslant 0}[ia_{i+j}q_{j+1,x}+a_{i+j}jq_{j+1,x}+a_{i+j,x}(j+1)q_{j+1}]. \tag{4.2b}$$

Next, recall [12] that moment-space systems of the form:

$$a_{i,t}=\sum_{j\geqslant 0}[ia_{i+j-1}R_j+a_{i+j}Q_j+a_{i+j,x}P_j]\qquad i\in\mathbb{Z}_+ \tag{4.3}$$

with R_j, Q_j, $P_j\in\mathscr{C}_a$, are lifted from the hydrodynamical-type systems

$$v_{,t}=\sum_{j\geqslant 0}\left[P_jv^jv_{,x}+R_jv^j-v_{,y}\int_0^y\mathrm{d}y(P_j(v^j)_{,x}+Q_jv^j)\right] \tag{4.4a}$$
$$\bar h_{,t}=\sum_{j\geqslant 0}[P_ja_{j,x}+Q_ja_j] \tag{4.4b}$$

via the moment map

$$a_i = \int_0^{\bar{h}} \mathrm{d}y\, v^i. \tag{4.5}$$

Also, systems of the form (4.3) and (4.4) form Lie algebras, and the map (4.5) is an isomorphism of these Lie algebras. It follows that all the hydrodynamical systems producing the modified Benney flows commute between themselves since so do the lifted modified Benney flows in the moment space (the first 'basic property'); from formulae (4.3) and (4.4) we see that our system (4.2*b*) is of the type (4.3) with

$$R_0 = 0 \qquad R_{j+1} = q_{j+1,x} \qquad Q_j = jq_{j+1,x} \qquad P_j = (j+1)q_{j+1} \tag{4.6}$$

so that the corresponding fluid system is

$$v_{,t} = \sum_{j\geq 0} \left\{ (q_{j+1}v^{j+1})_{,x} - v_{,y} \int_0^y \mathrm{d}y[(j+1)q_{j+1}(v^j)_{,x} + jq_{j+1,x}v^j] \right\} \tag{4.7a}$$

$$\bar{h}_{,t} = \sum_{j\geq 0} [(j+1)q_{j+1}a_{j,x} + jq_{j+1,x}a_j]. \tag{4.7b}$$

In particular, for $P = L^2/2$, we have $P_{\geq 1} = \frac{1}{2}\xi^2 + a_0\xi$, so that $q_2 = \frac{1}{2}$, $q_1 = a_0$, and the system (4.7) becomes the system (1.35).

The situation with kinetic representations is similar. Systems of the form [13]

$$f_{,t} = \sum_{j\geq 0} [P_j(p^j f)_{,x} - R_j(p^j f)_{,p} + Q_j(p^j f)] \qquad f = f(x, p, t) \tag{4.8}$$

where now the a_i are understood as

$$a_i = \int_{-\infty}^{\infty} \mathrm{d}p\, p^i f \qquad i \in \mathbb{Z}_+ \tag{4.9}$$

form a Lie algebra, and the moment map (4.9) is an isomorphism of Lie algebras taking the system (4.8) into the system (4.3). Again, the kinetic flows corresponding to the modified Benney equations (4.2*b*) commute between themselves and have, by formula (4.6), the form

$$f_{,t} = \sum_{j\geq 0} [(j+1)q_{j+1}p^j f_{,x} - q_{j+1,x}(p^{j+1}f)_{,p} + jq_{j+1,x}p^j f]. \tag{4.10}$$

Remark 4.1. In contrast to the unmodified case, neither of the systems (4.7) or (4.10) is Hamiltonian; equivalently, general Hamiltonian systems with the Hamiltonian structure b (1.32) cannot be lifted from a hydrodynamical or a kinetic system (since they are not of the form (4.3)). There remains a possibility that the second Hamiltonian structure of the modified Benney hierarchy, whose existence follows from the last remark in [12], produces flows of the hydrodynamical form (4.3), but this is difficult to establish.

5. Kinetic Miura map

In this section we prove formula (1.39).

Theorem 5.1. The map

$$F(x, p) = (p - a_0)f(x, p - a_0) \tag{5.1}$$

implies the map

$$A_i = \sum_{n\geqslant 0} \binom{i}{n} a_0^n a_{1+i-n} \tag{5.2}$$

for

$$A_i = \int_{-\infty}^{\infty} \mathrm{d}p\, p^i F \qquad i \in \mathbb{Z}_+ \tag{5.3}$$

$$a_i = \int_{-\infty}^{\infty} \mathrm{d}p\, p^i f \qquad i \in \mathbb{Z}_+ \tag{5.4}$$

Proof. We have

$$\begin{aligned} A_i &= \int_{-\infty}^{\infty} \mathrm{d}p\, p^i F && \text{(by (5.2)} \\ &= \int_{-\infty}^{\infty} \mathrm{d}p\, p^i (p-a_0) f(x, p-a_0) && \text{(changing } p \text{ into } p+a_0) \\ &= \int_{-\infty}^{\infty} \mathrm{d}p (p+a_0)^i pf = \int_{-\infty}^{\infty} \sum_{n\geqslant 0} \binom{i}{n} a_0^n p^{i-n} pf \\ &= \sum_{n\geqslant 0} \binom{i}{n} a_0^n a_{1+i-n}. \end{aligned} \tag{5.5}$$

Remark 5.2. There exists no simple map $\{v(x, y), \bar{h}(x)\} \mapsto \{u(x, y), h(x)\}$ which lifts up to the Miura map (5.2) in the moment spaces through the hydrodynamical lift

$$A_i = \int_0^h \mathrm{d}y\, u^i \qquad i \in \mathbb{Z}_+$$

$$a_i = \int_0^{\bar{h}} \mathrm{d}y\, v^i \qquad i \in \mathbb{Z}_+.$$

Remark 5.3. The full Miura map (1.28), before the quasiclassical limit is taken, can also be lifted from a single kinetic map whose quasiclassical limit is the map (5.2).

6. Combinatorial formulae

In this section we derive explicit formulae for the conserved densities of the modified Benney hierarchy.

Set

$$H_m := \frac{1}{m} \operatorname{Res}(L^m) \qquad m \in \mathbb{N} \tag{6.1}$$

$$H_0 := a_0. \tag{6.2}$$

Since

$$(L^m)_{,t} = \{P_{\geqslant 1}, L^m\}$$

and

$$\mathrm{Res}(\{\cdot,\cdot\}) \sim 0$$

and since

$$a_{0,t} \sim 0$$

by formula (2.5), we have

$$H_{s,t} \sim 0 \qquad \forall s \in \mathbb{Z}_+. \tag{6.3}$$

Theorem 6.1. The polynomials

$$H_m \in \mathbb{Q}[a_0, \ldots, a_m] \qquad m \in \mathbb{N}, \tag{6.4}$$

are uniquely defined by the formulae

$$\left.\frac{\partial^k H_m}{\partial a_{i_1} \ldots \partial a_{i_k}}\right|_{a=0} = \frac{(m-1)!}{(m-k)!}\,\delta\!\left(m+1, \sum_{\alpha=1}^{k} (i_\alpha+1)\right) \qquad H_m\Big|_{a=0} = 0 \tag{6.5}$$

where $\delta(\cdot,\cdot\cdot) := \delta_{\cdot}^{\cdot\cdot}$ is the Kronecker delta.

Proof. Since

$$[\mathrm{Res}(L^m)]|_{a=0} = \mathrm{Res}(\xi^m) = 0 \qquad \forall m \in \mathbb{N}$$

the uniqueness of the H_m follows from formulae (6.5).

Next

$$\frac{\partial H_m}{\partial a_i} = \frac{\partial}{\partial a_i}\left[\frac{1}{m}\,\mathrm{Res}(L^m)\right] = \mathrm{Res}(L^{m-1}\xi^{-i}) \tag{6.6}$$

so that

$$\begin{aligned}\left.\frac{\partial^k H_m}{\partial a_{i_1} \ldots \partial a_{i_k}}\right|_{a=0} &= \frac{m(m-1)\ldots(m-k+1)}{m}\,[\mathrm{Res}(L^{m-k}\xi^{-\Sigma i_\alpha})]\Big|_{a=0} \\ &= \frac{(m-1)!}{(m-k)!}\,\mathrm{Res}(\xi^{m-k-\Sigma i_\alpha}) = \frac{(m-1)!}{(m-k)!}\,\mathrm{Res}(\xi^{m-\Sigma(i_\alpha+1)}) \\ &= \frac{(m-1)!}{(m-k)!}\,\delta(m+1, \Sigma\,(i_\alpha+1))\end{aligned} \tag{6.7}$$

as required.

Remark 6.2. Formulae (6.5) show that

$$H_m \in \mathbb{Z}[a_0, \ldots, a_m] \tag{6.8}$$

and that

$$H_m \in a_m + \mathbb{Z}[a_0, \ldots, a_{m-1}]. \tag{6.9}$$

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References

[1] Wilson G 1981 On two constructions of conservation laws for Lax equations *Q. J. Math. Oxford (2)* **32** 491-512

[2] Manin Yu I 1978 Algebraic aspects of non-linear differential equations *Itogi Nauk. Tekhn. Ser. Sov. Probl. Matem.* **11** 5-152 (Engl. transl. 1979 *J. Sov. Math.* **11** 1-122)

[3] Kupershmidt B A 1985 Mathematics of dispersive water waves *Commun. Math. Phys.* **99** 51-73

[4] Lebedev D R and Manin Yu I 1979 Conservation laws and Lax representation of Benney's long wave equations *Phys. Lett.* **74A** 154-6

[5] Kupershmidt B A and Manin Yu I 1977 Long-wave equations with free boundaries. I. Conservation laws *Funct. Anal. Appl.* **11** (3) 31-42

[6] Kupershmidt B A and Manin Yu I 1978 Equations of long waves with a free surface. II. Hamiltonian structure and higher equations *Funct. Anal. Appl.* **12** (1) 25-37

[7] Benney D J 1975 Some properties of long nonlinear waves *Stud. Appl. Math.* **52** 45-50

[8] Zakharov V E 1980 Benney equations and quasiclassical approximation in the method of the inverse problem *Funct. Anal. Appl.* **14** (2) 15-24

[9] Gibbons J 1983 Collisionless Boltzmann equations and integrable moment equations *Physica* **3D** 503-11

[10] Kupershmidt B A and Wilson G 1981 Modifying Lax equations and the second Hamiltonian structure *Invent. Math.* **62** 403-36

[11] Kupershmidt B A 1983 Deformations of integrable systems *Proc. R. Irish Acad.* A **83** 45-74

[12] Kupershmidt B A 1984 Normal and universal forms in integrable hydrodynamical systems *Proc. NASA Ames-Berkeley 1983 Conf. on Nonlinear Problems in Optimal Control and Hydrodynamics* ed R L Hunt and C Martin (Brookline, MA: Math. Sci. Press) pp 357-78

[13] Gibbons J and Kupershmidt B A 1990 Connections between free-surface hydrodynamical systems and kinetic equations *Phys. Lett.* A, in press